# IRS of Polish groups (with a focus on $S_\infty$)

w.i.p. with Matthieu Joseph

## 1) IRS of locally compact groups - what is differert with Polish groups

$\Gamma$ a locally compact group ; $Sub(\Gamma)$ = "closed subgp of $\Gamma$"

$Sub(\Gamma)$ is a compact set in the ~~[illegible]~~ Chabauty topology

$\Gamma \curvearrowright Sub(\Gamma)$ continuously by conjugacy.

Def

An invariant random subgroup is an invariant mesure on $Sub(\Gamma)$.

Examples

1) Normal subgroups
(this is precisely the notion IRSs are trying to generalize)

2) $\Gamma \curvearrowright (X, \mu)$ pmp

we have a ~~continuous~~ measurable map $Stab : X \to Sub(\Gamma)$
$x \mapsto Stab_\Gamma(x)$

$$\nu = Stab_* \mu \text{ is an IRS.}$$

Theorem (Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet)

All IRS are produced this way

Idea: $X = Cos(\Gamma) = \{gH : g \in \Gamma, H \in Sub(\Gamma)\}$

Using Haar measure, ~~one~~ can construct a measure on $Cos(\Gamma)$ that is pushed to a given IRS.

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$G$ a Polish group (no Haar measure!)

$Sub(G)$ is not compact, but it is Borel for the Effros Borel structure.

If $G \curvearrowright (X, \mu)$ ~~then~~ $Stab : X \to Sub(G)$ is analytic (thus $\mu$ measurable)

... we cannot (yet?) say much more

($Cos(G)$ is also standard Borel)

. A special Polish group $S_\infty = Perm(\mathbb{N})$

$\sigma, \tau \in S_\infty$

$$d(\sigma, \tau) = \frac{1}{1 + \min\{n : \sigma(n) \neq \tau(n)\}}$$

$+ \min(n : \sigma^{-1}(n) \neq \tau^{-1}(n))$

. A basis for the topology is

$$U_{x_1, \dots, x_n, y_1, \dots, y_n} = \{g \in S_\infty \text{ st } g(x_i) = y_i\}$$

Exercise: it is not locally compact.

Rk: $S_\infty$ is generally rich in dynamical properties; it is amenable; any action of $S_\infty$ on a compact that is minimal admits a unique invariant measure.

$S_\infty \curvearrowright [0,1]^{\mathbb{N}}$ (De Finetti): all inv. measures are of the form $\nu^{\mathbb{N}}$.

- Subgroups of $S_\infty$

To understand subgroups of $S_\infty$, we need to understand <u>structures</u>

Definition

~~L is a Lang~~ We call (relational) Language a collection $\mathcal{L}$ of symbols to each of which is associated a number

Example $\mathcal{L} = (R_1, R_2, R_3)$

$R_1$ of arity 1

$R_2$ of arity 2

$R_3$ of arity 1.

A structure in a Language is 1) a domain

2) an interpretation of the relation in $\mathcal{L}$

$$M = (A, R_1^M, R_2^M, R_3^M)$$

$$R_1^M \subset A \quad R_2^M \subset A^2 \quad R_3^M \subset A^3$$

- Example: $\mathcal{L} = \{E\}$ of arity 2

a graph is an $\mathcal{L}$ structure.

Theorem (~~KPT~~ Folklore?)

Any closed subgroup of $S_\infty$ is $Aut(M)$ where $M$ is a structure with domain $\mathbb{N}$.

Proof $\mathcal{L} = \{R_{x_1, \dots, x_n} : x_1, \dots, x_n \in \mathbb{N}\}$

$G \leqslant_{closed} S_\infty$ we associate $M = (\mathbb{N}, (R^M_{x_1,\dots,x_n}))$

where $\cancel{R_{x_1,\dots,x_n}}$ $y_1, \dots, y_n \in R^M_{x_1, \dots, x_n} \iff \exists g \in G$ st

$g x_1, \dots, x_n = y$

. IRS of $S_\infty$

Thm (J.-Joseph)
Any IRS of $S_\infty$ (or any of its closed subgroup) can be realized as the stabilizer of an action.

Proof Idea: push the IRS via the map defined in the previous proof □ → needs to check things

Rk: ~~$\mathrm{Cos}(S_\infty) \simeq$ Struc~~
~~but the usual identification is not G-equivariant.~~

Q: Are there any interesting IRS of $S_\infty$?
Yes!

Thm (Ackerman-Freer-Patel)
Let $\mathbb{A}$ be a structure that satisfies some general condition, then there are measures on Struc that concentrate on the ~~isomorphic~~ structures isomorphic to $\mathbb{A}$.

If you push that measure by Aut, you get an IRS concentrated on the $S_\infty$ orbit of Aut($\mathbb{A}$).

. There even seem to be much more
Q: What IRS of Aut($\mathbb{A}$) for a given $\mathbb{A}$?
↳ We do not know.

Another path: An IRS induces a partition of $\mathbb{N}$
Kingman: We understand very well random partitions of $\mathbb{N}$
↕
This is very strongly connected to satisfying a De Finetti
Thm J. Tsankov: There are a lot of Sub($S_\infty$) that do.